Mathematics Methods: Units 3 & 4
Test 3: Logarithms
Calculator-Free Section

Time allowed: 20 minutes
Total marks: 35

Formula sheet provided
No notes permitted
No ClassPad (nor any other calculator) permitted

Name: MARKING KEY   Teacher (circle): MARTIN   SMITH   MOORE

Note: You should show clear and comprehensive working out throughout to obtain part marks where these apply.

1. Evaluate the following expressions giving your answers in the simplest form.

[1 + 1 + 2 + 2 = 6 marks]

(a) $\log_6 36$

$= \log_6 6^2$

$= 2 \cdot \log_6 6$

$= 2$ ✓

(b) $\log_3 \frac{1}{27}$

$= \log_3 3^{-3}$

$= -3 \cdot \log_3 3$

$= -3$ ✓

(c) $\log_9 \sqrt{3}$

$= \log_9 \sqrt{\sqrt{9}}$ ✓

$= \log_9 9^{1/4}$

$= \frac{1}{4} \log_9 9$

$= \frac{1}{4}$ ✓

(d) $5^{3\log_5 2}$

$= (5^{\log_5 2})^3$ ✓

$= 2^3$

$= 8$ ✓

2. Express each of the following as a single logarithm. Simplify your answers where possible.

[2 + 2 + 2 = 6 marks]

(a) $3\log_{10} x + 5\log_{10} y$

$= \log_{10} x^3 + \log_{10} y^5$ ✓ indices

$= \log_{10} (x^3 y^5)$ ✓

(b) $\log_2 \frac{1}{x} - \log_2 \frac{1}{x^2} + 3$

$= \log_2 \left(\frac{1}{x} \cdot \frac{x^2}{1}\right) + 3$

$= \log_2 x + 3$ ✓ terms with x.

$= \log_2 x + \log_2 8$

$= \log_2 (8x)$ ✓

(c) $\ln(x^2 - 1) - \ln(x + 1)$

$= \ln\left[\frac{x^2-1}{x+1}\right]$ ✓

$= \ln\left[\frac{(\cancel{x+1})(x-1)}{(\cancel{x+1})}\right]$

$= \ln(x-1)$ ✓

12

3. Find all possible values of x satisfying the following equations. Where your answers involve logarithms, express these using natural logarithms.

[2 + 2 + 3 = 7 marks]

(a) $\log_3(3x-3) = 2$

$\log_3(3x-3) = \log_3 9$ ✓

$3x-3 = 9$

$x = 4$ ✓

(b) $7^{1-x} = 6^x$

$\ln 7^{1-x} = \ln 6^x$

$(1-x)\ln 7 = x\ln 6$. ✓ log of both sides and log law.

$x(\ln 6 + \ln 7) = \ln 7$

$x = \dfrac{\ln 7}{\ln 6 + \ln 7}$ ✓

$= \dfrac{\ln 7}{\ln 42}$ (better).

(c) $\log_{10} x + \log_{10}(x-21) = 2$

$\log_{10}[x(x-21)] = \log_{10} 100$

$x(x-21) = 100$ ✓

$x^2 - 21x - 100 = 0$

$(x-25)(x+4) = 0$ ✓ factorises

$x = 25$ ✓ positive sol^n only.

4. Determine f′(x) for each of the following functions. Simplify your answers where possible, and where your answers involve logarithms express these using natural logarithms.

[2 + 2 + 3 + 3 = 10 marks]

(a) $f(x) = 3x^2 + 2\ln x$

$f'(x) = 6x + \dfrac{2}{x}$ ✓ ✓

(b) $f(x) = \ln[(x+2)(x-5)]$

$= \ln[x^2 - 3x - 10]$ ✓ expand or log law or product rule

$f'(x) = \dfrac{2x-3}{x^2-3x-10}$ ✓ answer (single fraction)

or $= \dfrac{2x-3}{(x+2)(x-5)}$

(c) $f(x) = e^{2x}\log_2 x$

$= \dfrac{1}{\ln 2} \cdot e^{2x} \cdot \ln x$ ✓ change of base

$f'(x) = \dfrac{1}{\ln 2}\left(e^{2x} \cdot \dfrac{1}{x} + 2e^{2x} \cdot \ln x\right)$ ✓ product rule

$= \dfrac{e^{2x}}{\ln 2}\left(\dfrac{1}{x} + 2\ln x\right)$ ✓ answer ~~(factors out e^{2x})~~

(d) $f(x) = \ln\left[\dfrac{x}{1-x}\right]$

$= \ln x - \ln(1-x)$ ✓ log law or quotient rule

$f'(x) = \dfrac{1}{x} - \dfrac{-1}{1-x}$ ✓

$= \dfrac{(1-x)+x}{x(1-x)}$

$= \dfrac{1}{x(1-x)}$ ✓ answer (single fraction)

max (-1) overall for answers as sum of two algebraic fractions.

17

5. Evaluate the following indefinite integrals. (Assume that the domains are restricted to ensure that the denominators in any fractions are greater then zero.)

[2 + 2 + 2 = 6 marks]

(a) $\int \left(3x^2 + \frac{4}{x}\right)dx$

$= x^3 + 4\ln x + c$ ✓ ✓

{• accept answers w/ absolute value here.
• +C missing ⇒ (-1) overall}

(b) $\int \frac{4x - 10}{x^2 - 5x}\,dx$

$= 2\int \frac{2x-5}{x^2-5x}\,dx.$

✓ recognises derivative in numerator

$= 2\ln(x^2-5x) + c$ ✓

(c) $\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}\,dx$

$= -\frac{1}{2}\int \frac{+2\cos 2x - 2\sin 2x}{\sin 2x + \cos 2x}\,dx.$

✓ recognises derivative in numerator

$= -\frac{1}{2}\ln(\sin 2x + \cos 2x) + c.$ ✓

[END OF SECTION]

Mathematics Methods: Units 3 & 4
Test 3: Logarithms
Calculator-Assumed Section

SHENTON COLLEGE

Time allowed: 30 minutes
Total marks: 27

Formula sheet provided
1 single-sided A4 page of notes permitted
ClassPad (and/or other calculator) permitted

Name: MARKING KEY. Teacher (circle): MARTIN SMITH MOORE

Note: You should show clear and comprehensive working out throughout to obtain part marks where these apply.

1. The ear is sensitive to a very wide range of sound intensities. As such, the perceived loudness of a sound is measured on a logarithmic scale in units called decibels (dB). The loudness of a sound of intensity I is given by

$$L = 10\log\frac{I}{I_0}$$

where $I_0 = 10^{-12}$ W/m² is a reference intensity defined as that of a barely audible sound.

[2 + 3 = 5 marks]

(a) Find the loudness, to the nearest decibel, of a hairdryer with a sound intensity of 1.58×10^{-5} W/m².

$L = 10\log\left(\dfrac{1.58\times10^{-5}}{10^{-12}}\right)$ ✓ substitutes correctly

$= 72$ dB ✓ answer (units, rounding)

(b) A normal conversation has a loudness of 50 dB. Sitting in the front row at a rock concert has a loudness of 110 dB. How many times greater is the intensity of sound at the rock concert compared with that of the normal conversation?

$L_2 - L_1 = 10\log(I_2/I_0) - 10\log(I_1/I_0)$

$= 10\log(I_2/I_1)$

$110 - 50 = 10\log(I_2/I_1)$

$\dfrac{I_2}{I_1} = 10^6$ i.e., it is a million times greater.

✓ log law
✓ substitute
✓ rearrange and answer.

OR.
✓ $I_1 = 10^{-1}$ W/m²
✓ $I_2 = 10^{-7}$ W/m²
ratio.

5

2. The rate at which a battery charges becomes slower the closer the battery gets to its maximum charge C_0. The time (in hours) taken for a completely flat battery to be charged to a charge C is

$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery.

[3 + 2 = 5 marks]

(a) Rearrange the equation above to give an equation showing how the charge on an initially flat battery changes as a function of time. (I.e., rearrange it to the form C=)

$-t/k = \ln(1 - C/C_0)$ ✓ divide by -k

$e^{-t/k} = 1 - C/C_0$ ✓ applies inverse function.

$C/C_0 = 1 - e^{-t/k}$

$C = C_0(1 - e^{-t/k})$ ✓

(b) For a certain battery k=0.25. How long will it take for this battery to charge to 95% of its maximum charge? Give your answer to two decimal places.

$t = -0.25 \cdot \ln\left(1 - \frac{0.95\cancel{C_0}}{\cancel{C_0}}\right)$

$= -0.25 \cdot \ln 0.05$

$= 0.75$ hours

accept sub. into exponential form.

✓ substitutes correctly

✓ answer (units, d.p.)

(44.94 mins)

max. -1 for units/rounding between Q1, Q2.

3. Consider the curve defined by

$$y = \frac{\ln x}{\sqrt{x}}$$

In this question, give all of your answers using exact values.

[4 + 1 + 4 = 9 marks]

(a) Show that this curve has a local maximum and give the exact value of its coordinates.

$$y' = \frac{2 - \ln x}{2\sqrt{x^3}}$$ ✓derivative

$$y' = 0 \Rightarrow 2 - \ln x = 0$$
$$x = e^2$$ ✓ single stationary point.

$$y'' = \frac{3\ln x - 8}{4\sqrt{x^5}} \qquad y''\big|_{x=e^2} = -\frac{1}{2e^5} < 0 \Rightarrow \text{local max.}$$ ✓ second derivative test or sign test.

$$y\big|_{x=e^2} = \frac{2}{e} \qquad \left(e^2, 2/e\right)$$ ✓coordinates

(b) Determine the equation of the tangent to this curve at the point (1,0).

$$y = x - 1$$ ✓

-1 overall for decimal answers.

(c) Find the coordinates of the point of intersection of the tangent found in Part (b) and the tangent to the curve at its local maximum.

$$y = x - 1 \qquad y = 2/e.$$ ✓ tangent at local max.

$$x - 1 = 2/e$$ ✓equate lines.

$$x = 1 + 2/e.$$ ✓ solve.

so the point of intersection is $\left(1 + 2/e, 2/e\right)$ ✓coordinates.

9

4. Consider the following two logarithmic functions:

$$f(x) = \ln x \quad \text{and} \quad g(x) = \ln(x-2) + 2$$

[2 + 1 + 1 + 4 = 8 marks]

(a) State the equations of any asymptotes for these two functions.

$f(x)$ has $x = 0$ ✓ ; $g(x)$ has $x = 2$. ✓

(don't require them to say which is which)

(b) Determine the exact value of the x-coordinate of their point of intersection.

$\ln x = \ln(x-2) + 2$. ~~✓equate~~

$x = \dfrac{2e^2}{e^2 - 1}$ ✓ answer.

(−1 overall for decimal answers in (b) and (c)).

(c) Determine the exact x-value of the root of g(x).

$0 = \ln(x-2) + 2$. ✓ ~~set = 0~~

$x = 2 + \dfrac{1}{e^2}$ ✓ answer.

(d) Determine, to two decimal places, the area trapped between the two curves and the x-axis.

$$A = \int_1^{2e^2/e^2-1} \ln x \, dx - \int_{2+1/e^2}^{2e^2/e^2-1} (\ln(x-2)+2)\, dx.$$

$= 0.54$

✓ limits on first $\int$

✓ limits on second $\int$

✓ minus sign (difference).

✓ answer (rounding).

didn't deduct mark for extra d.p.

[END OF TEST]

8